Solution Booklet

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gratefully received

143 problem proposals submitted by 12 countries:

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Contents

I-1

Let $\mathbb{R}^+$ be the set of positive real numbers. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$ it holds that

$$yf^{2025}(x) \geq xf(y)\,.$$

Show that there exists a positive integer n_0 such that for all positive integers $n \geq n_0$ and for all $x \in \mathbb{R}^+$ it holds that

$$f^n(x) \geq x.$$

Remark. Here f^n denotes the function f applied n times, this means $f^n(x) = \underbrace{f(f(\ldots f}_{n \text{ times}}(x)\ldots))$.

Solution. By assumption, we know that $xf(y) \leq yf^{2025}(x)$ for all $x, y \in \mathbb{R}^+$. Plugging in $y = x$, we can deduce that $f(x) \leq f^{2025}(x)$ for all $x \in \mathbb{R}^+$.

We claim that there is some $y \in \mathbb{R}^+$ with $y \leq f(y)$. To see this, recall that $f(1) \leq f^{2025}(1)$. Therefore, there must be some $i \in \{1, 2, \ldots, 2024\}$ such that $f^i(1) \leq f^{i+1}(1)$. Taking $y = f^i(1)$ shows that there indeed exists some $y \in \mathbb{R}^+$ with $y \leq f(y)$.

Now, taking $y \in \mathbb{R}^+$ with $y \leq f(y)$ and any $x \in \mathbb{R}^+$, note that we have

$$xf(y) \leq yf^{2025}(x) \leq f(y)f^{2025}(x),$$

implying $x \leq f^{2025}(x)$. Thus, we have $x \leq f^{2025}(x)$ for all $x \in \mathbb{R}^+$.

So, we have shown that for any $x \in \mathbb{R}^+$ we have $x \leq f^{2025}(x)$ and $f(x) \leq f^{2025}(x)$. In particular, applying this to $f^m(x)$ and to $f^{m-1}(x)$ for any integer $m \geq 2$, we obtain $f^m(x) \leq f^{m+2025}(x)$ and $f^m(x) \leq f^{m+2024}(x)$. Applying this repeatedly, we can therefore conclude that $f^m(x) \leq f^{m+a\cdot 2024+b\cdot 2025}(x)$ for any $x \in \mathbb{R}^+$, any integer $m \geq 2$, and any non-negative integers a and b.

Finally, we claim that for any $n \geq 2025^2$ and any $x \in \mathbb{R}^+$ we have $x \leq f^n(x)$. Note that $n - 2025 \geq 2024 \cdot 2025$, and so (as 2024 and 2025 are coprime) we can write $n - 2025 = a \cdot 2024 + b \cdot 2025$ for some non-negative integers a and b. Thus, we can conclude that

$$x \leq f^{2025}(x) \leq f^{2025+a\cdot 2024+b\cdot 2025}(x) = f^n(x),$$

as desired.

I-2

On an infinite square grid, on which some unit squares are coloured red, a *ruby rook* is a piece which, in one move, can travel any number of squares in one direction parallel to one of the grid lines (either vertically or horizontally), while remaining on red squares at all times throughout the move.

Starting with an uncoloured infinite square grid, Alice performs the following procedure: First, she colours at most 2025 of the unit squares red. Afterwards, she places some ruby rooks on distinct red unit squares, such that the following two rules are satisfied:

- No ruby rook can reach another ruby rook in one move.
- Every ruby rook can reach every other ruby rook in two moves.

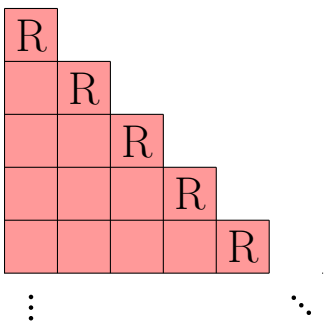
Find the maximum possible number of ruby rooks that Alice can place during this procedure.

Answer. The maximum number of ruby rooks that Alice can place is 63.

Proof that 63 **ruby rooks can be placed.** Consider a staircase-shaped red area made up of 63 rows ranging from length 1 to 63 (see the figure below). It is possible to place 63 ruby rooks along the main diagonal of the red shape since:

- No ruby rook can reach another ruby rook in one move since there is at most one ruby rook in every row and in every column.
- Every ruby rook can reach every other ruby rook within two moves. Indeed, it can first move horizontally to the left to a square in the same column as the other ruby rook, and then move vertically upwards to the other ruby rook or first move vertically downwards to a square in the same row as the other ruby rook and then horizontally to the right to the second ruby rook.

The number of red squares is $\frac{63\cdot(63+1)}{2} = 2016 \leq 2025$, satisfying the condition in the problem statement.



Proof that at most 63 **ruby rooks can be placed.** Assume that Alice places n ruby rooks, and let us enumerate them from 1 to n. For any two ruby rooks $1 \leq i < j \leq n$ on the grid, there must exist a red square that both ruby rook i and ruby rook j can reach in one move, and let us call this red square $S_{i,j}$ (if such a square did not exist, then it would not be possible for ruby rook i to get to ruby rook j in two moves). Note that if $(i,j) \neq (i',j')$, then $S_{i,j}$ and $S_{i',j'}$ are distinct; otherwise, we would have three ruby rooks that can reach the same square, which would imply that we have a pair of ruby rooks in the same row or column that can reach each other in one move. Note that none of the n red squares with a ruby rook can coincide with any square $S_{i,j}$, and the squares $S_{i,j}$ are also all red. Since the number of squares $S_{i,j}$ is $\binom{n}{2}$, we get

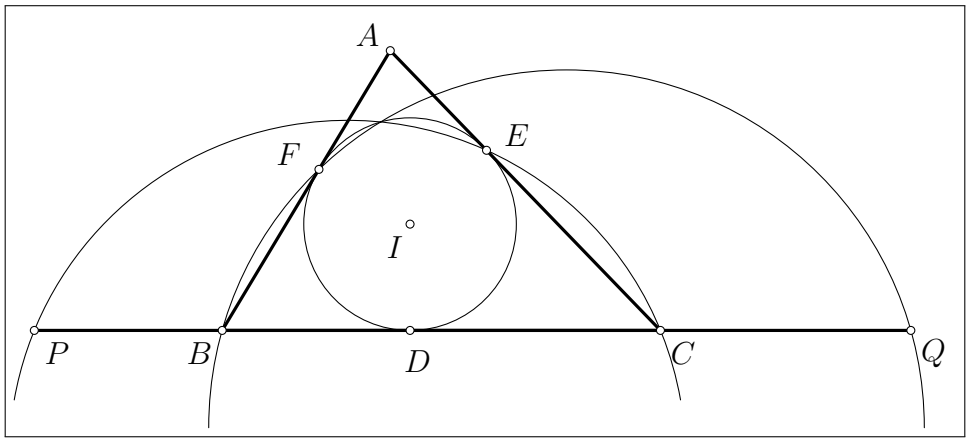
$$\binom{n}{2} + n \leq 2025.$$

This implies $(n+\frac{1}{2})^2 - \frac{1}{4} = n(n+1) = n(n-1) + 2n = 2 \cdot \binom{n}{2} + 2n \leq 4050 \leq 4096 - \frac{1}{4}$. Thus, $(n+\frac{1}{2})^2 \leq 4096 = 64^2$, and we can conclude that $n + \frac{1}{2} \leq 64$. Since the number n of ruby rooks is an integer, this implies $n \leq 63$.

I-3

Let ABC be a triangle. Its incircle ω touches the sides BC, CA and AB at points D, E and F, respectively. Let P and Q be points on the line BC distinct from D such that $PB = BD$ and $QC = CD$. Prove that the circumcircles of the triangles PCE and QBF and the circle ω pass through a common point.

Solution 1.



We denote the interior angles in the triangle ABC by $\alpha := \measuredangle BAC$, $\beta := \measuredangle CBA$ and $\gamma := \measuredangle ACB$.

Let I_A be the center of the excircle opposite to the vertex A. By the definition of P and because BE and BD are tangent to the incircle of ABC we have $BP = BD = BE$. Therefore the triangle BFP is isosceles with base PF and

$$\measuredangle PFB = \measuredangle BPF = \frac{180^\circ - \measuredangle FBP}{2} = \frac{\beta}{2} \quad .$$

In the same way, it follows in triangle CEQ that $CQ = CE$ and

$$\measuredangle CEQ = \measuredangle EQC = \frac{180^\circ - \measuredangle QCE}{2} = \frac{\gamma}{2} \quad .$$

Claim 1. $PFEQ$ are cyclic.
To this end we consider the chord FQ and show $\measuredangle QPF + \measuredangle FEQ = 180^\circ$. We have shown that $\measuredangle QPF = \beta/2$. On the other hand we obtain

$$\begin{aligned} \measuredangle FEQ &= \measuredangle FED + \measuredangle DEC + \measuredangle CEQ = \measuredangle FDB + (90^\circ - \gamma/2) + \gamma/2 \\ &= (90^\circ - \beta/2) + 90^\circ = 180^\circ - \beta/2 \quad , \end{aligned}$$

which proves the claim. ■

Claim 2. I_A is the circumcenter of $PFEQ$.
The triangles BFP and CQE are isosceles. Therefore, the perpendicular bisectors of of the line segments PF and QE are the bisectors of the exterior angles at B and C, respectively. On the one hand these two lines intersect in the circumcenter of $PFEQ$, on the other hand they intersect in I_A. Hence, I_A is the circumcenter of $PFEQ$. ■

Claim 3. I_APCE are cyclic (and analogously I_ABFQ).
We consider the chord PE. Obviously, we have $\measuredangle ECP = \gamma$. From the central angle - inscribed angle theorem in $PFEQ$ we obtain

$$\measuredangle EI_AP = 2\measuredangle EQP = 2 \cdot \gamma/2 = \gamma \quad ,$$

i.e. $\measuredangle EI_AP = \measuredangle ECP$ and it follows claim 3. ■

Let $G \neq D$ be the intersection of line I_AD and the incircle of ABC. We show that G is the desired common point.

Claim 4. I_AFGQ are cyclic (and analogously I_AEGP).
We consider the chord FI_A. From the tangent chord angle theorem in the incircle of ABC, it follows

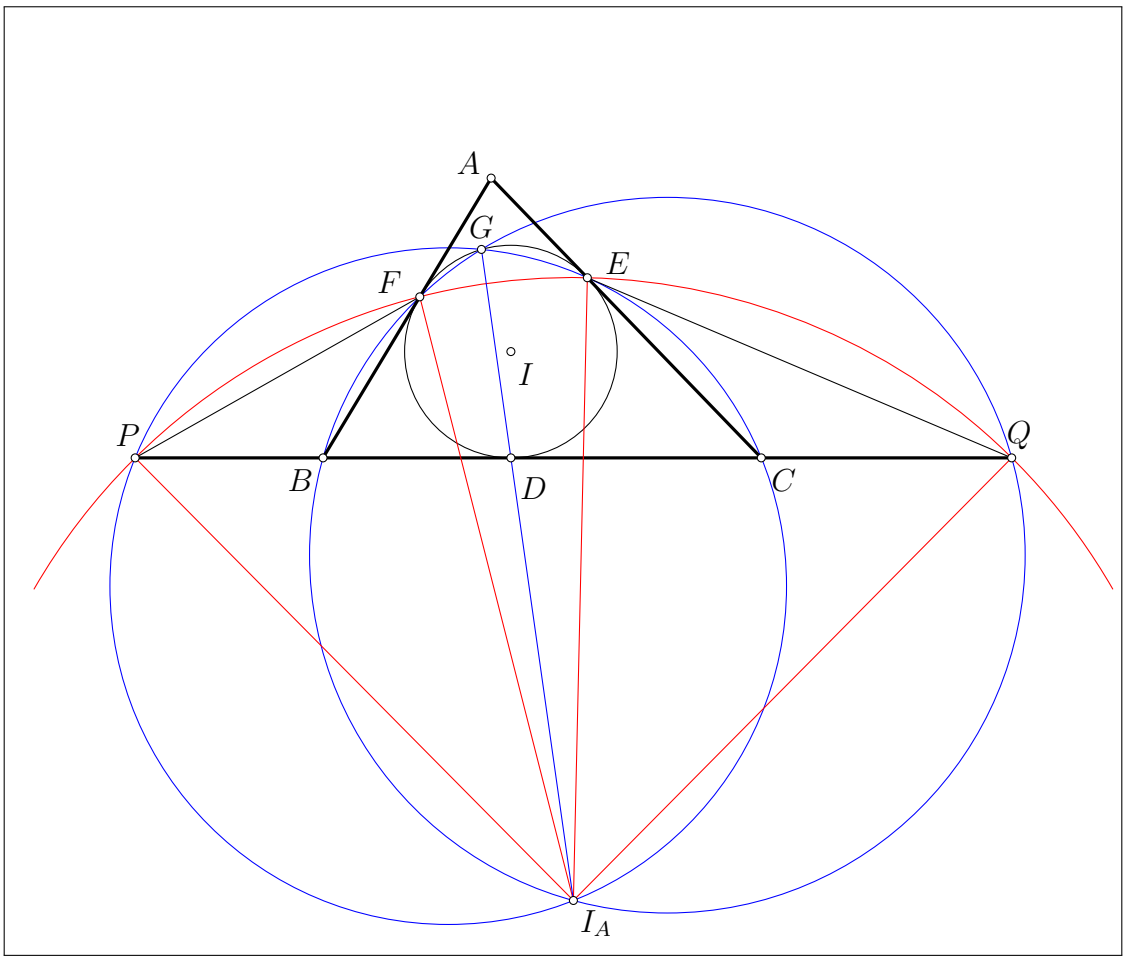
$$\measuredangle FGI_A = \measuredangle FGD = \measuredangle BFD = 90^\circ - \beta/2 \quad .$$

Since FQI_A is isosceles and from the central angle - inscribed angle theorem in $PFEQ$ we have

$$\measuredangle FQI_A = 90^\circ - \frac{\measuredangle QI_AF}{2} = 90^\circ - \frac{2\measuredangle QPF}{2} = 90 - \beta/2 \quad ,$$

i.e. $\measuredangle FGI_A = \measuredangle FQI_A$ and I_AFGQ is cyclic. ■

The statement follows directly from claim 4. □



Solution 2. We denote by I_A the center of the excircle opposite to the vertex A and by D_A, E_A and F_A the points of tangency of this excircle with the lines BC, CA and AB, respectively. As usual we use the notations $a = BC$, $b = CA$ and $c = AB$, $s = \frac{a+b+c}{2}$, $\alpha = \measuredangle BAC$, $\beta = \measuredangle CBA$, $\gamma = \measuredangle ACB$ and r_A for the radius of the excircle opposite to A.

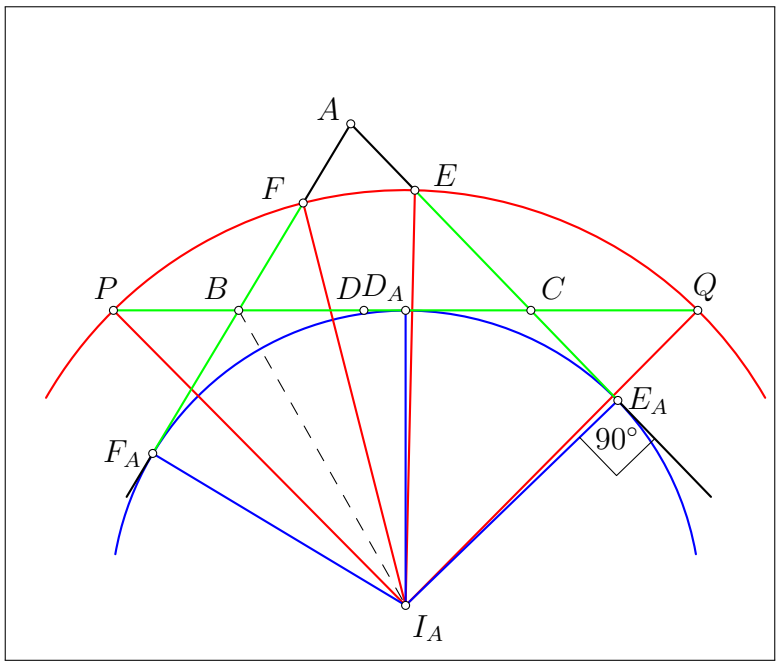
Claim 1. $PFEQ$ is cyclic with circumcenter I_A.
By considering the tangent segments from the incircle and the excircle, we obtain the following relations:

$$AE = AF = s - a \quad , \quad BP = BD = BF = CD_A = s - b \quad ,$$
$$CQ = CD = CE = BD_A = s - c \quad , \quad AE_A = AF_A = s \quad .$$

It follows that $PD_A = PB + BD_A = s - b + s - c = a$, $D_AQ = D_AC + CQ = s - b + s - c = a$, $E_AE = AE_A - AE = s - (s - a) = a$ and $F_AF = AF_A - AF = s - (s - a) = a$. We conclude that the rectangular triangles I_AD_AP, I_AD_AQ, I_AE_AE and I_AF_AF are congruent with

$$(I_AP)^2 = (I_AF)^2 = (I_AE)^2 = (I_AQ)^2 = r_A^2 + a^2 \quad ,$$

which shows our claim. ■



Claim 2. BI_AQF is cyclic (and analogously PI_ACE).
We show that $\measuredangle FI_AB = \measuredangle FQB$ which will prove the claim. From the central angle - inscribed angle theorem in $PQEF$ we obtain

$$\measuredangle FI_AP = 2\measuredangle FQP \quad .$$

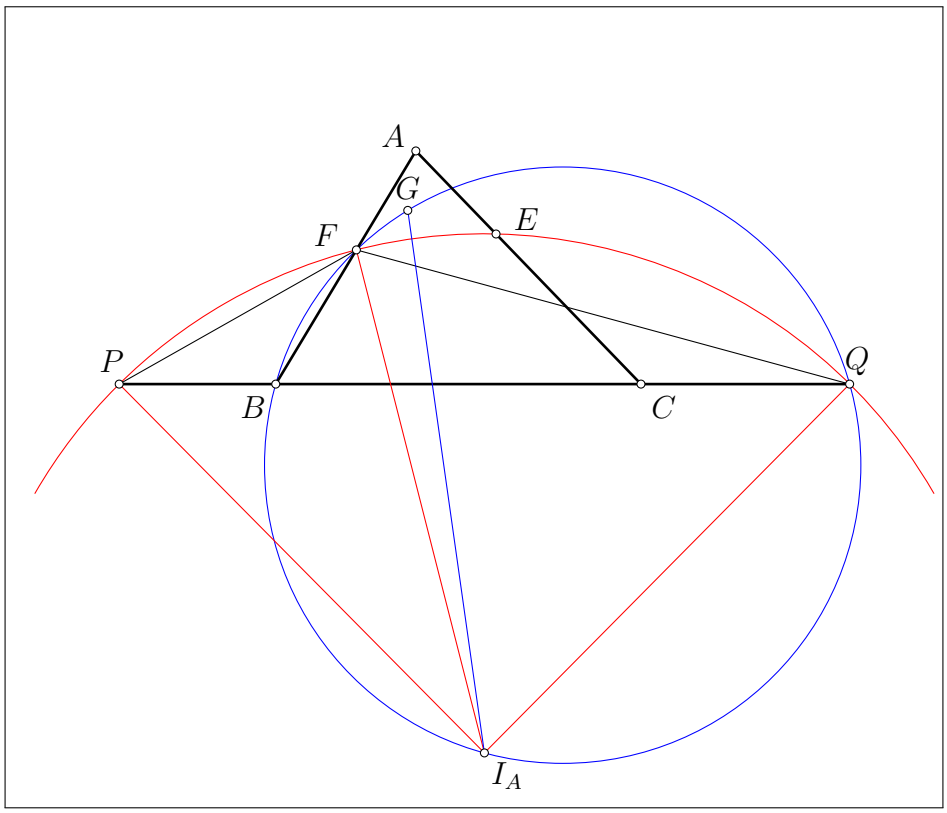
On the other hand the triangles BI_AF and BI_AP are congruent, because they have the side BI_A in common, $FI_A = PI_A$ and $\measuredangle BFI_A = \measuredangle I_APB$ (the last equation follows from the congruence of the triangles I_AF_AF and I_AD_AP). Therefore, we have $\measuredangle FI_AP = 2\measuredangle FI_AB$ and finally $\measuredangle FI_AB = \measuredangle FQB$. ■

We denote the intersection of the circumcircles of PI_ACE and BI_AQF by G.

Claim 3. $\measuredangle FGI_A = 90° - \beta/2$ (and analogously $\measuredangle I_AGE = 90° - \gamma/2$).
In the cyclic quadrilateral FI_AQG we have $\measuredangle FGI_A = \measuredangle FQI_A$. In the isosceles triangle FI_AQ we obtain $\measuredangle FQI_A = 90° - \measuredangle QI_AF/2$. From the central angle - inscribed angle theorem in $PQEF$ we get $\measuredangle QI_AF = 2\measuredangle QPF$. In the isosceles triangle PBF with exterior angle β at B we have $\measuredangle QPF = \measuredangle BPF\beta/2$. Taken all together we get

$$\measuredangle FGI_A = \measuredangle FQI_A = 90° - \measuredangle QI_AF/2 = 90° - \frac{2\measuredangle QPF}{2} = 90° - \beta/2 \ .$$

■



claim 4. $DEGF$ is cylic.
From claim 3 we have

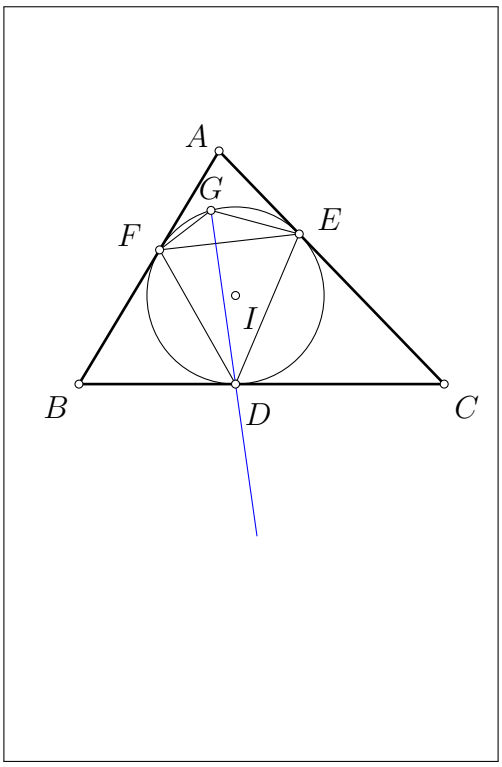
$$\measuredangle FGE = \measuredangle FGI_A + \measuredangle I_AGE = 90° - \beta/2 + 90° - \gamma/2 = 180° - \beta/2 - \gamma/2 \quad .$$

On the other hand in triangle DEF we obtain

$$\measuredangle EDF = 180° - \measuredangle FDB - \measuredangle CDE = 180° - (90° - \beta/2) - (90° - \gamma/2) = \beta/2 + \gamma/2 \quad ,$$

i.e. $\measuredangle FGE + \measuredangle EDF = 180°$, which proves the claim. ■

The statement follows directly from claim 4.



Solution 3. Let $E' \neq E$ be the second intersection of the line ED with the circumcircle of $\triangle PCE$, and let $F' \neq F$ be the second intersection of the line FD with the circumcircle of $\triangle QBF$. Then we have

$$|DE| \cdot |DE'| = |DP| \cdot |DC| = 2 \cdot |DB| \cdot |DC| = |DB| \cdot |DQ| = |DF| \cdot |DF'|,$$

so $EFE'F'$ is a cyclic quadrilateral. Thus, $\measuredangle F'E'E = \measuredangle F'FE = \measuredangle DFE = \measuredangle CDE$ (where at the last step we used that CD is tangent to the incircle of $\triangle ABC$ through D, E and F). Therefore the line $E'F'$ is parallel to the line BC.

Furthermore, since $CEPE'$ is a cyclic quadrilateral by the definition of E', and since $\triangle CED$ is an isosceles triangle with base DE, we have

$$\measuredangle E'PD = \measuredangle E'PC = \measuredangle E'EC = \measuredangle DEC = \measuredangle CDE = \measuredangle PDE'.$$

This shows that $\triangle DPE'$ is isosceles with base PD. As B is the midpoint of DP, this we can conclude that $\measuredangle E'BC = \measuredangle E'BD = 90°$.

Analogously we can show that that $\triangle QDF'$ is isosceles with base QD and $\measuredangle BCF' = 90°$.

Combining the facts $\measuredangle E'BC = 90°$ and $\measuredangle BCF' = 90°$ and $E'F' || BC$, we can conclude that $E'F'CB$ is a rectangle. Let S be the reflection of D upon the midpoint of this rectangle. Then

S lies on the segment $E'F'$ and satisfies $|E'S| = |DC|$ and $|SF'| = |BD|$. Then $E'SCD$ and $SF'DB$ are parallelograms.

Using $E'S||PC$ and $SC||E'D$ and the fact that $\triangle DPE'$ is isosceles with base PD, we can conclude that $E'SCP$ is an isosclese trapezoid. In particular, S is on the circle through E', C and P (which also contains E). This implies that the five points E', S, C,E and P are all on a common circle. Analogously we can show that the five points F', Q, F,B and S are all on a common circle. We concude, that S is one intersection of these two circles. Let $G \neq S$ be the second intersection of these two circles. We need to show that G lies on the incircle of $\triangle ABC$

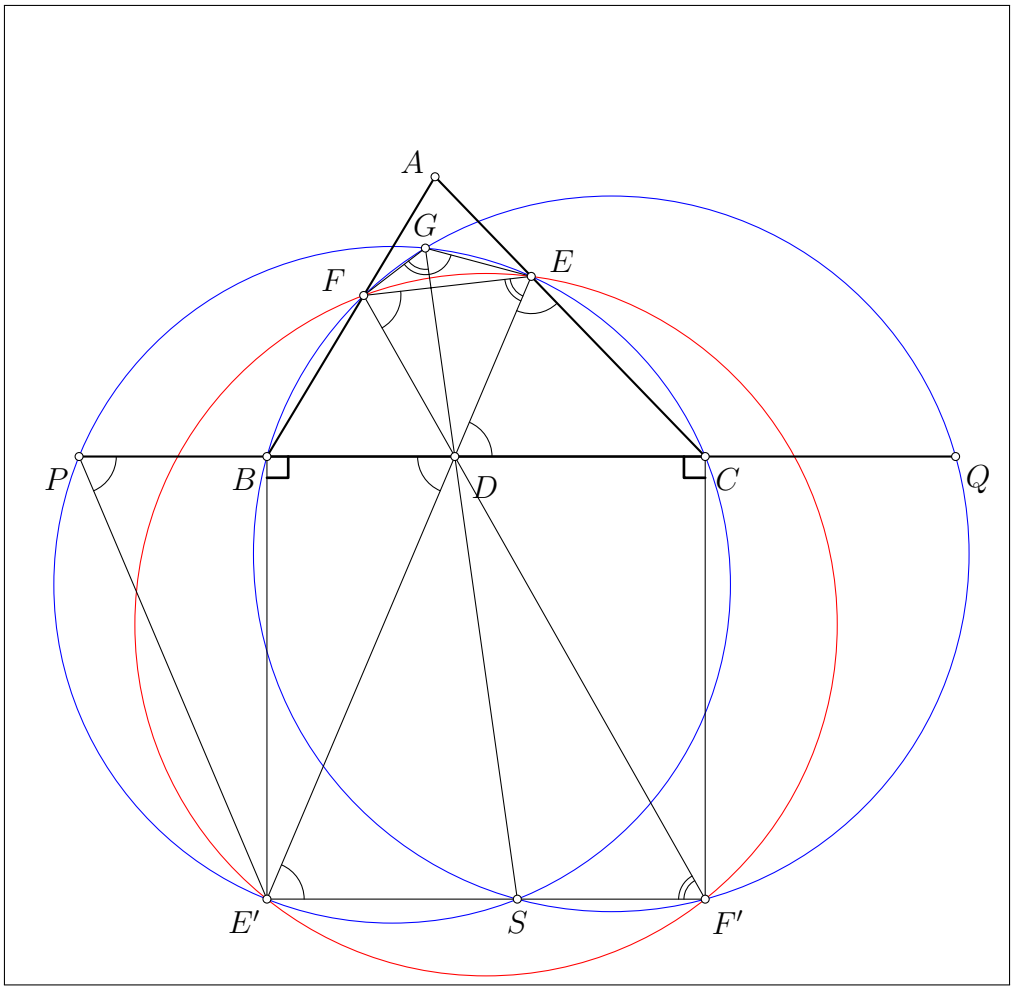
Now, we have (recalling that $EFE'F'$ is a cyclic quadrilateral)

$$\measuredangle SGE = \measuredangle SE'E = \measuredangle F'E'E = \measuredangle F'FE = \measuredangle DFE.$$

Analogously, we can show $\measuredangle FGS = \measuredangle FED$. Adding both of these equations, we obtain

$$\measuredangle FGE = \measuredangle FGS + \measuredangle SGE = \measuredangle FED + \measuredangle DFE = 180° - \measuredangle EDF.$$

This shows that $DEGF$ is a cyclic quadrilateral, so G lies on the incircle of $\triangle ABC$ through D, E and F.



Solution 4.

Claim. *Let $PF \cap QE = L$. Then $L \in \omega$.*

Proof. Denote the angles of $ABC\Delta$ by α, β, γ as standard. By the equalities $BD = BF = BP$ and $CD = CE = CQ$, we have:

- $\angle DPF = \frac{\beta}{2}$,
- $\angle EQD = \frac{\gamma}{2}$,
- hence $\angle FLE = 180° - \frac{\beta+\gamma}{2}$.
- $\angle FDB = 90° - \frac{\beta}{2}$,
- $\angle EDC = 90° - \frac{\gamma}{2}$,
- hence $\angle EDF = 180° - \angle FDB - \angle EDC = \frac{\beta+\gamma}{2}$.

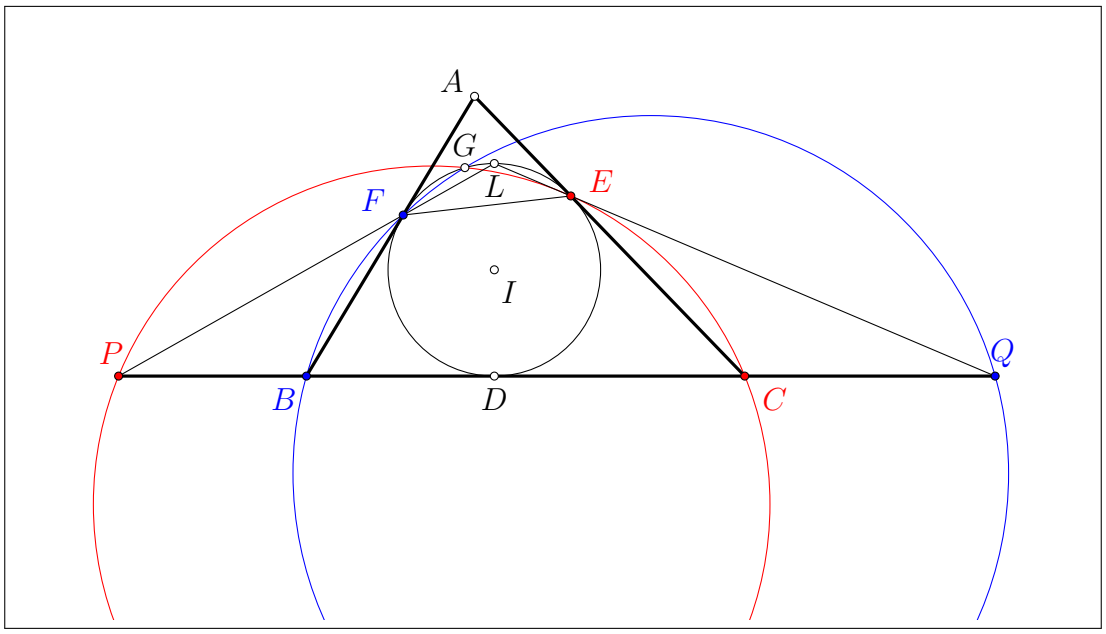
So $\angle FLE + \angle EDF = 180°$, meaning that $FLED$ is cyclic, showing the claim. □

Now let G be the centre of the spiral similarity taking FE to PQ. It is well-known that $G = (LFE) \cap (LPQ) = \omega \cap (LPQ)$.

Claim. *The points G, P, C, E are concyclic.*

Proof. Using the similarity of triangles GFE and GPQ, we have $\angle CPG + \angle GEC = \angle EFG + (\angle FEC + \angle GEF) = \angle FEC + (180° - \angle FGE) = (180° - \angle AEF) + \angle EDF = 90° + \frac{\alpha}{2} + \frac{\beta+\gamma}{2} = 180°$. □

Similarly, G, B, F, Q are concyclic, implying that (PCE) and (BQF) meet at G, on ω.



I-4

A subset S of the integers is called *Saxonian* if for every three pairwise different elements a, b, $c \in S$ the number $ab + c$ is the square of an integer. Prove that any Saxonian set is finite. Determine the largest possible number of elements that a Saxonian set can have.

Answer. Every Saxonian set has at most 3 elements, and there exist Saxonian sets consisting of exactly 3 elements.

Solution. The set $\{0, 1, 4\}$ is an example of a Saxonian set with 3 elements, as is shown by an easy calculation.

Thus, it remains to prove the following claim.

Claim: A Saxonian set can not have 4 or more elements.

Proof: Suppose, to the contrary, that there is such set S, and let $a < b < c < d$ be some of its elements. We distinguish between two cases.

First case. $c \geq 0$.

Then one also has $d > 0$. If $a < 0$ then $ad + c \leq (-d) + c < 0$ can not be a perfect square. Therefore, it follows that, in fact, $d > c > b > a \geq 0$.

Since S is Saxonian, $cd + a = k^2$ and $cd + b = l^2$, for some nonnegative integers k and l. Moreover, $b > a$ implies that $l \geq k + 1$, and by plugging in this inequality again, one finds

$$cd + b = l^2 \geq (k + 1)^2 = k^2 + 2k + 1 = cd + a + 2k + 1\,.$$

This yields

$$k \leq \frac{b - a - 1}{2}\,. \tag{1}$$

On the other hand, $d > c > 0$ and $a \geq 0$ together imply

$$k^2 = cd + a \geq cd > c^2\,,$$

which shows $k > c$.

But this leads to a contradiction, as inequality (1) implies $k \leq (b-a-1)/2 < b/2 < b < c$.

Second case. $c < 0$.

Then one also has $a < 0$. If $d \geq 0$ then $ad + c \leq (-d) + c < 0$ can not be a perfect square. Therefore, it follows that $a < b < c < d < 0$.

Let us write $a =: -A$, $b =: -B$, $c =: -C$, and $d =: -D$. Then $0 < D < C < B < A$.

Since S is Saxonian, $ab + c = AB - C = k^2$ and $ab + d = AB - D = l^2$, for some nonnegative integers k and l. Moreover, $D < C$ implies that $l \geq k + 1$, and by plugging in this inequality again, one finds

$$AB - D = l^2 \geq (k+1)^2 = k^2 + 2k + 1 = AB - C + 2k + 1\,.$$

This yields

$$k \leq \frac{C - D - 1}{2}\,. \tag{2}$$

On the other hand, $B > C > 0$ and $A > B$ together imply

$$k^2 = AB - C > AB - B = (A-1)B \geq B^2\,,$$

which shows $k > B$.

But this is impossible, as inequality (2) implies $k \leq (C - D - 1)/2 < C/2 < C < B$.

Thus, both cases above turned out to be contradictory. Therefore, a Saxonian set of at least four elements does indeed not exist.

Remark: There are many Saxonian sets of three elements. A computer experiment delivers already 107 Saxonian sets $\{a, b, c\}$ of integers $-100 \leq a < b < c \leq 100$. Among them, there are very obvious ones such as those of type $\{0, i^2, j^2\}$, for positive integers $i \neq j$, but as well many others, such as $\{-98, -59, -6\}$.

T-1

Bob has n coins with integer values

$$c_1 \geq c_2 \geq \cdots \geq c_n > 0\,.$$

He is standing in front of a vending machine that offers n candy bars with positive integer costs $b_1, b_2, \ldots, b_n$. Bob notices that for every $i \in \{1, \ldots, n\}$, it holds that

$$b_1 + b_2 + \cdots + b_i \geq c_1 + c_2 + \cdots + c_i.$$

Furthermore, the total value of Bob's coins equals the sum of the costs of all the candy bars. The candy bars can be purchased in any order. In order to buy the i-th candy bar, Bob has to insert coins of total value at least b_i. However, the machine does not give him back any change.

Prove that Bob can buy at least half of the candy bars.

Solution 1. Let $m \coloneqq \lfloor \frac{n}{2} \rfloor$ and let

$$S \coloneqq b_1 + b_2 + \cdots + b_n = c_1 + c_2 + \cdots + c_n.$$

We call the coins with value $c_1, \ldots, c_m$ the *valuable coins*. In the following, we will show that Pepa can buy the bars that cost $b_{m+1}, \ldots, b_n$.

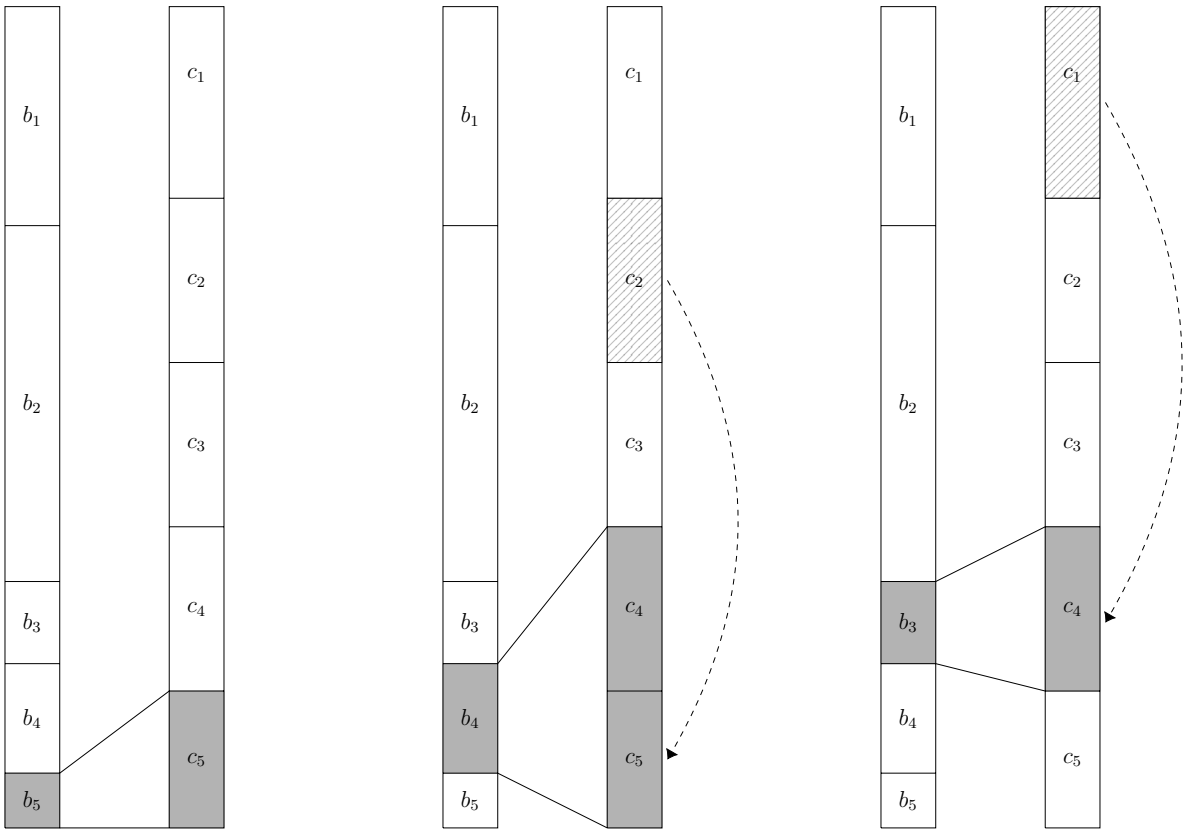
Rewriting the condition from the problem statement for $i = m$ yields

$$\begin{aligned} & b_1 + b_2 + \cdots + b_m \geq c_1 + c_2 + \cdots + c_m, \\ \iff \quad & S - b_{m+1} - b_{m+2} - \cdots - b_n \geq S - c_{m+1} - c_{m+2} + \cdots + c_n, \\ \iff \quad & c_{m+1} + c_{m+2} + \cdots + c_n \geq b_{m+1} + b_{m+2} + \cdots + b_n. \qquad (\dagger) \end{aligned}$$

We can represent the situation visually by splitting an interval of length S into n segments $B_1, \ldots, B_n$ of length $b_1, \ldots, b_n$ in that order, and another interval of length S into n segments $C_1, \ldots, C_n$ of length $c_1, \ldots, c_n$ in that order. We view these two intervals side by side.

We would like to pay for the bar represented by B_i (where $i > m$) using the coins represented by those segments that overlap the interval B_i. That would clearly be enough to pay for the bar, but some coins might be used multiple times to pay for different bars. Luckily, because of (†), we have not used any valuable coins at all. We just need to show that we can use these valuable coins in place of the coins that were used multiple times.

When did we use the coin represented by C_j to pay for both the bars represented by B_i and B_{i+1}? If and only if the boundary between B_i and B_{i+1} lies inside the interval C_j. Hence, the



number of times we use the coin represented by C_j equals one plus the number of boundaries inside of C_j. We are only buying $n-m$ bars and thus the number of boundaries between them is at most $n-m-1 \leq m$. (Some boundaries between the B_i's might align perfectly with boundaries between the C_j's. In that case, the number of repeated uses of some coin will be even smaller than $n-m-1$.)

This means that we have enough valuable coins to cover for the coins used multiple times. Each of these valuable coins is at least as valuable as any of the "original" coins that were used multiple times, so we successfully pay for the chosen candy bars. □

Solution 2. We may assume without loss of generality that $b_1 \geq \cdots \geq b_n$. Indeed, if this were not the case and π would be the permutation such that $b_{\pi(1)} \geq \cdots \geq b_{\pi(n)}$, then for every $i \in \{1, \ldots, n\}$, we would have

$$b_{\pi(1)} + \cdots + b_{\pi(i)} \geq b_1 + \cdots + b_i \geq c_1 + \cdots + c_i.$$

Thus, the problem requirement is also fulfilled for $b_{\pi(1)}, \ldots, b_{\pi(n)}$ instead of $b_1, \ldots, b_n$.

Claim. *The following holds:*

(a) $c_n \geq b_n$;

(b) $c_{n-2k+1} + c_{n-2k} \geq b_{n-k}$, *for each* $k \in \{1, \ldots, \lceil n/2 \rceil - 1\}$.

Once we prove the claim, we are done because it explicitly shows how Pepe can buy the candy bars which cost $b_{\lfloor n/2\rfloor+1}, \ldots, b_n$.

Proof of the claim. First, we derive the inequality

$$c_\ell + c_{\ell+1} + \cdots + c_n \geq b_\ell + b_{\ell+1} + \cdots + b_n \tag{$*$}$$

for any $\ell \in \{1, \ldots, n\}$ in essentially the same way as in the first solution, where we did this only for $\ell = m+1$. Taking $(*)$ for $\ell = n$ immediately justifies (a).

Now, because the sequences $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ both are weakly decreasing, for any $k \in \{1, \ldots, \lceil n/2\rceil - 1\}$, we have

$$\begin{aligned} c_{n-2k} + c_{n-2k+1} &\overset{(1)}{\geq} \frac{2}{2k+1}\left(c_{n-2k} + c_{n-2k+1} + \cdots + c_n\right) \overset{(*)}{\geq} \frac{2}{2k+1}\left(b_{n-2k} + b_{n-2k+1} + \cdots + b_n\right) \\ &\overset{(2)}{\geq} \frac{2}{2k+1}\left(b_{n-2k} + b_{n-2k+1} + \cdots + b_{n-k}\right) \overset{(3)}{\geq} \frac{2}{2k+1} \cdot (k+1) \cdot b_{n-k}. \end{aligned}$$

These inequalities are justified as follows: (1) holds because the average of c_{n-2k}, c_{n-2k+1} is clearly at least as large as the average of $c_{n-2k}, \ldots, c_n$, (2) is merely throwing away some non-negative terms, and (3) is bounding the smallest element of $b_{n-2k}, \ldots, b_{n-k}$ by the average. Overall, we obtain

$$c_{n-2k} + c_{n-2k+1} \geq \frac{2k+2}{2k+1} \cdot b_{n-k} > b_{n-k}. \qquad \square$$

T-2

Let $\mathbb{R}^+$ be the set of positive real numbers. Determine all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all numbers $x, y \in \mathbb{R}^+$, we have

$$f(xy) + f(x) = f(y)f\big(xf(y)\big) + f(x)f(y)\,,$$

and there exists at most one number $a \in \mathbb{R}^+$ such that $f(a) = 1$.

Answer. The function $f(x) = 1/x$ is the only function solving the functional equation.

Proof that this is a solution. For the function $f(x) = 1/x$, we indeed have

$$f(xy) + f(x) = \frac{1}{xy} + \frac{1}{x} = \frac{1}{y} \cdot \frac{1}{x/y} + \frac{1}{x} \cdot \frac{1}{y} = f(y)f(xf(y)) + f(x)f(y)$$

for all $x, y \in \mathbb{R}^+$. The equality $f(a) = 1$ is only attained for $f(a) = \frac{1}{a} = 1 \Leftrightarrow a = 1$.

Proof that there are no other solutions. The solution consists of two parts. In the first (more complicated) part, we show that $f(1) = 1$ follows from the functional equation, so 1 is actually in the image of f. In the second part, we show then that $f(x) = \frac{1}{x}$ is the only function satisfying the functional equation where 1 is the unique preimage of 1 (i.e. where $f(1) = 1$ but $f(a) \neq 1$ for all $a \neq 1$). For simplicity of notation, let in both parts $P(x, y)$ denote the equality from the problem and denote $f(f(x))$ by $ff(x)$.

First part. We want to describe as much dependencies as possible in terms of 1 and $f(1)$ to finally show $f(1) = 1$. The key insights to this part are not being afraid of complicated constants and looking at a simplified version of $P(x, yf(1))$ later.

Plugging in easy examples for x and y gives us

$$P(1,1) \implies ff(1) = 2 - f(1) \tag{1}$$

$$P(x,1) \overset{(1)}{\implies} f(f(1)x) = \frac{2 - f(1)}{f(1)} f(x) = \frac{ff(1)}{f(1)} f(x) \tag{2}$$

$$P(x, f(1)) \overset{(1),(2)}{\implies} f(ff(1)x) = \frac{2 - 2f(1) + f(1)^2}{f(1)ff(1)} f(x) \tag{3}$$

$$(2),(3) \implies f\Big(\frac{ff(1)}{f(1)}x\Big) = \frac{2 - 2f(1) + f(1)^2}{f(1)ff(1)} f\Big(\frac{x}{f(1)}\Big) = \frac{2 - 2f(1) + f(1)^2}{ff(1)^2} f(x). \tag{4}$$

This allows us to simplify $ff(1)$ and $f(cx)$ for some constants c. Note that $f(x) > 0$, so all denominators are positive.

Now,

$$\begin{aligned} P(x, yf(1)) &\overset{(2)}{\Longrightarrow} \quad f(x) + f(f(1)xy) = f(f(1)y)f\Big(\frac{ff(1)}{f(1)}xf(y)\Big) + f(x)f(f(1)y) \\ &\overset{(2),(4)}{\Longrightarrow} \quad f(x) + \frac{ff(1)}{f(1)}f(xy) = \frac{2 - 2f(1) + f(1)^2}{f(1)ff(1)}f(y)f(xf(y)) + \frac{ff(1)}{f(1)}f(x)f(y), \end{aligned} \tag{5}$$

using (2) and (4) to pull out the constant factors.

The identity (5) is similar to $P(x, y)$ with different coefficients, so we may subtract a multiple of $P(x, y)$ to anihilate the term $f(y)f(xf(y))$ and get

$$\begin{aligned} &\Big(1 - \frac{2 - 2f(1) + f(1)^2}{f(1)ff(1)}\Big)f(x) + c \cdot f(xy) = c \cdot f(x)f(y) \\ \overset{(1)}{\Longrightarrow} \quad &\frac{-2(1 - f(1))^2}{f(1)(2 - f(1))}f(x) = -c \cdot f(xy) + c \cdot f(x)f(y), \end{aligned} \tag{6}$$

where c is a constant we do not need to calculate.

As the right side of (6) does not change when we exchange x and y and the left side is constant in y, we can conclude that the left side of (6) must also be also constant in x. So $f(1) = 1$ and we are done with the first part, or we get that f is constant. But as $2 = f(1) + ff(1)$ by (1), this also implies $f(1) = 1$.

Second part. We now know that $f(1) = 1$ and use it without further notice.

First, we note that the right side of the functional equation only depends on x and $f(y)$, but not on y, so for $f(y_1) = f(y_2)$, we get

$$f(x) + f(xy_1) = f(x) + f(xy_2).$$

By choosing $x = \frac{1}{y_1}$, this implies that $f(\frac{y_2}{y_1}) = f(1) = 1$. By the assumption that there is at most one a with $f(a) = 1$, this implies $y_1 = y_2$, so f is injective.

Now,

$$\begin{aligned} P(1, y) &\implies \quad 1 = f(y)ff(y) \\ P(1, f(y)) &\implies \quad 1 = ff(y)fff(y) \end{aligned} \tag{7}$$

so $f(y) = \frac{1}{ff(y)} = fff(y)$. By injectivity, we can conclude $y = ff(y)$. So, (7) implies that $f(y) = \frac{1}{ff(y)} = \frac{1}{y}$ for all y. In other words, the function f must be given by $f(x) = 1/x$ for all $x \in \mathbb{R}^+$.

Second part, alternative. We have $f(1) = 1$ and use it without further notice. Moreover, we have

$$P(1, y) \implies 1 = f(y)ff(y) \tag{7}$$

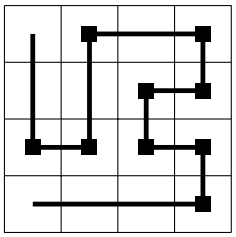
$$P(f(x), f(x)) \overset{(7)}{\implies} f(f(x)^2) = (ff(x))^2 \tag{8}$$

$$P(f(x), x) \overset{(7),(8)}{\implies} f(xf(x)) = 1 \tag{9}$$

As 1 is the only preimage of the value 1 for the function f, from (9) we can deduce $xf(x) = 1$. So we have $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R}^+$.

T-3

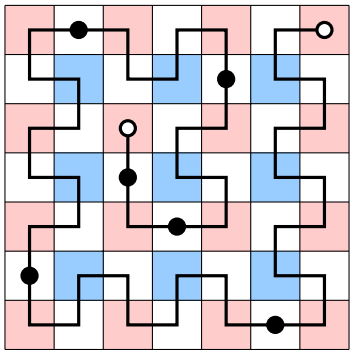
A *snake* in an $n \times n$ grid is a path composed of straight line segments between centres of adjacent cells, going through the centres of all the n^2 grid cells, which visits each cell exactly once. Here two grid cells are considered to be adjacent if they share an edge. Note that all pieces of the snake path are parallel to grid lines. The figure shows an example of a snake in a 4×4 grid. This snake makes nine 90° turns, marked by small black squares.



Let us now consider a snake through the 2025 cells of a 45×45 grid. What is the maximum possible number of 90° turns that such a snake can make?

Answer. The maximum number of 90° turns is 1979.

Optimal Construction. The maximum number 1979 if turns can be achieved by the "double snail". The following figure shows these for a 7×7 grid.



Such a double snail goes around the board in L-shaped layers of width 2, as is depicted in the figure. There are two cells in each layer it passes straight through. The only other non-corners are the starting and ending point, which gives $45^2 - 44 - 2 = 1979$ turns.

Proof of that at most 1979 turns are possible. Color the table as shown in the figure above: color like in chess, but instead of black, color alternately by red and blue, with red in the corners of the table. This gives 23^2 red and 22^2 blue cells, so there are 45 more red cells than blue cells. If the snake turns in a white cell, the cells immediately before and after are of different colors (one is red and one is blue). Thus, when considering the sequence of colored (red or blue) cells along the snake, between any two consecutive red cells in this sequence, the snake must pass through a white cell without turning there. The sequence of colored cells consists of 23^2 red cells and 22^2 blue cells, so in this sequence there must be at least $23^2 - 22^2 - 1 = 2 \cdot 22 = 44$ pairs of consecutive red cells. Therefore, there must be at least 44 white cells on which the snake is not turning. Together with the two endpoints of the snake (which must be on colored

cells since the snake is alternating between colored cells and white cells, and there are more colored than white cells), we find at least 46 cells where the snake does not make a 90° turn. Thus, we can have at most $45^2 - 46 = 1979$ turns, as claimed.

Alternative proof that at most 1979 turns are possible. Consider a snake in a 45×45 grid. We call a cell *straight* if the snake does not have a 90° turn in that cell. In particular, the first and the last cell of the snake are straight. We will show that the snake has at least 46 straight cells. This implies that the number of 90° turn of the snake is at most $45^2 - 46 = 2025 - 46 = 1979$.

Claim 1. Each row and each column of the grid contains at least one straight cell.

Proof. Indeed, consider any row R, and suppose that R does not contain a straight cell. In particular, the start and end of the snake cannot be in R. When traversing the snake from start to end, the snake will enter R vertically from the top or bottom, turn to take a horizontal step along R to a neighboring cell, and then turn away vertically again. Later, the snake will enter R again, take another horizontal step along R to a neighboring cell, and then leave R again; and so on. In particular, there would need to be equally many cells in R, where the snake enters R and where the snake exits R. However, this is impossible, since R contains 45 cells, which is an odd number.

For columns, the proof is similar. □

This already shows the existence of at least 45 straight cells.

Now, let us assume without loss of generality that the first step of the snake (from the starting point to its second cell) is horizontal (otherwise we can consider a reflection of the snake along the main diagonal of the grid). Let the first two cells of the snake be in row R.

Claim 2. R contains another straight cell besides the first cell of the snake.

Once we have proved the claim, we are done since R contains at least 2 straight cells, and each of the other 44 rows contains at least one straight cell (making a total of at least 46 straight cells).

Proof of Claim 2. Suppose that R does not contain another straight cell besides the start of the snake. In particular, the end of the snake cannot be in R. When traversing the snake from start to end, the snake will start in R, take a horizontal step along R to a neighboring cell, and then turn away vertically. Afterwards, the snake would enter vertically, take a horizontal step along R to a neighboring cell, and then exit R again there. In particular, there would need to be equally many cells in R, where the snake starts or enters R and where the snake exits R. However, this is impossible, since R contains 45 cells, which is an odd number. □

Comments. The problem (and the solutions) can be generalized in a straightforward way for any odd n in place of 45, the maximum possible number of turns is then $n^2 - n - 1$.

For even n, the "double snail" construction gives the value $n^2 - n$, but we do not have a proof of its optimality.

T-4

Let n be a positive integer. In the province of Laplandia there are $100n$ cities, each two connected by a direct road, and each of these roads has a toll station collecting a positive amount of toll revenue. For each road, the revenue of its toll station is split equally between the two cities at the ends of the road (meaning that each of the two cities receives half of the income). For each city, the total toll revenue is given by the sum of the revenues it receives from the $100n - 1$ toll stations on its roads.

According to a new law, the revenues of some of the toll stations will be collected by the federal government instead of by the adjacent cities. The governor of Laplandia is allowed to choose those toll stations. The mayors of the cities demand that for each city, the sum of the remaining revenues it receives from the other toll stations after this change is at least 99% of its former total toll revenue.

Find the largest positive integer k, depending on n, such that the governor can always choose k toll stations for the federal government to collect the toll revenue, while satisfying the demand of the city mayors.

Answer. $k = 100\binom{n}{2}$.

We consider a complete graph whose vertex set is the set of the $100n$ cities in Laplandia. We assign weights to the edges, where for every edge the weight is the revenue from the toll station of the corresponding road that each of the two cities at its endpoints receives (i.e. half of the total revenue of the road). The question now asks how many edges can be removed such that at each vertex the sum of the weights of the incident edges decreases by at most 1%.

Proof of $k \leq 100\binom{n}{2}$. Let us consider an assignment of edge weights, where the weights on all of the edges are equal (meaning that each road collects the same amount of toll revenue). In this case, one cannot remove more than 1% of the $100n-1$ edges incident to any vertex. Thus, at every vertex at most $\lfloor \frac{1}{100}(100n-1) \rfloor = n-1$ edges can be removed. So by the handshake lemma, the total number of edges that can be removed is at most $\frac{1}{2}(n-1)(100n) = 100\binom{n}{2}$.

Proof of $k \geq 100\binom{n}{2}$. Now consider any assignment of positive edge weights. We need to show that we can remove $100\binom{n}{2}$ edges in such a way that at every vertex the sum of the edge weights is decreased by at most 1%.

Partition the vertices of the graph into 100 (not necessarily equally sized) parts to minimize the sum of the weights of all edges whose endpoints lie in the same part of the partition. By the minimality of this choice, we have the following: For any vertex v in part P and any other part P', the sum of the weights of the edges going from v to P' is at least as large as the sum of the weights of the edges going from v to the other vertices in P (otherwise we could move

v from P to P' and improve our partition). Therefore, for each vertex v, the edges from v to the other vertices in P contribute at most a fraction of $\frac{1}{100} = 1\%$ of the total sum of the weights at vertex v. Thus, we can remove all edges whose end vertices belong to the same part in the partition, and this will decrease the sum of the weights at every vertex by at most 1%. It remains to show that when doing this, we remove at least $\binom{n}{2}$ edges.

Letting $a_1, a_2, \ldots, a_{100}$ be the sizes of the parts of the vertex partition (where $a_1 + \cdots + a_{100} = 100n$), the number of removed edges is

$$\begin{aligned}\binom{a_1}{2} + \binom{a_2}{2} + \cdots + \binom{a_{100}}{2} &= \sum_{i=1}^{100} \frac{1}{2}(a_i^2 - a_i) = \frac{1}{2}(a_1^2 + a_2^2 + \cdots + a_{100}^2) - 50n \\ &\geq 50n^2 - 50n = 100\binom{n}{2},\end{aligned}$$

where the inequality follows from the inequality between arithmetic and quadratic mean.

T-5

Let ABC be an acute triangle with $AB < AC$. Denote by D the foot of the perpendicular from A to BC. Let E be the point such that $ABEC$ is a parallelogram. Let M be a point inside triangle ABC such that $MB = MC$. Let F be the reflection of point D across the tangent to the circumcircle of triangle ADM at point M. Prove that $AF = DE$.

Solution 1. Let D' be the projection of E on BC. The parallelogram $ABEC$ is point symmetric with respect to the center of the line segment BC. Therefore, D and D' are also symmetric with respect to this center and we obtain $DE = AD'$. Hence we only need to prove $AD' = AF$.

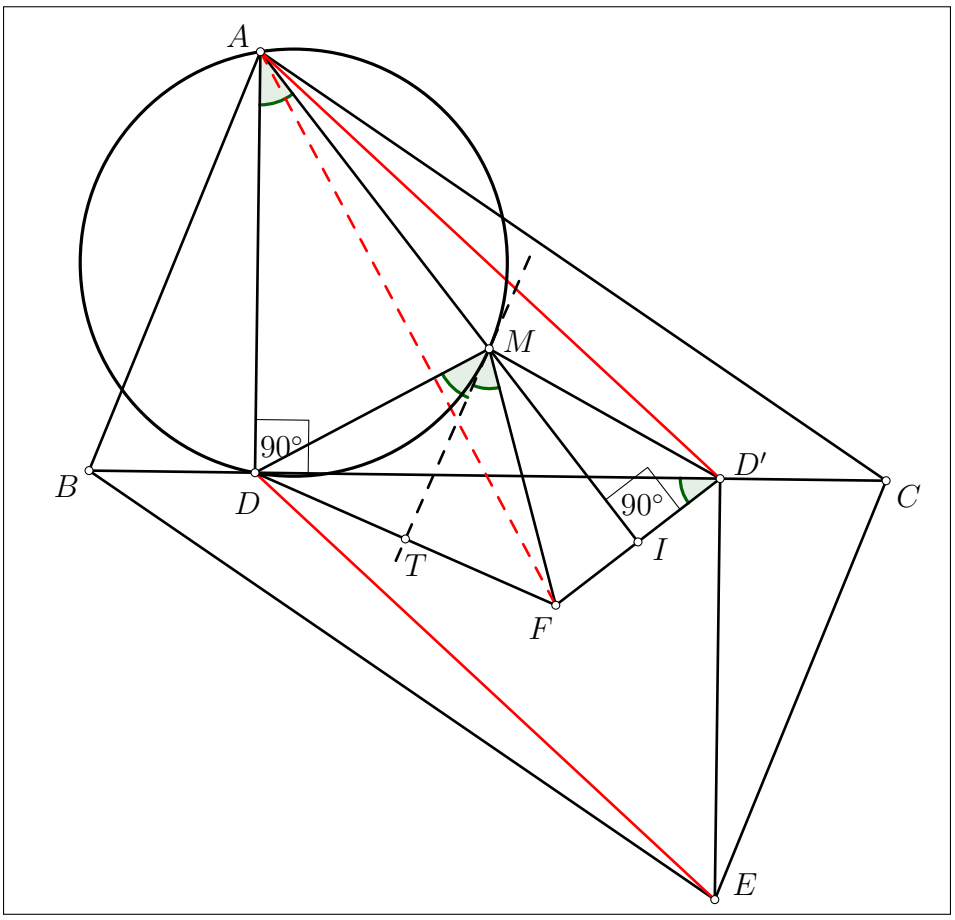
Let T be the center of the line segment DF. By definition of the point F, the line TM is the tangent to the circumcircle of ADM at M and $MD = MF$. In addition we have $MD = MD'$ since the point M is on the perpendicular bisector of BC and D and D' are symmetric with respect to this bisector. It follows that M is the circumcenter of triangle DFD'.

From the tangent chord angle theorem with chord DM in the circumcircle of ADM we obtain $\measuredangle DAM = \measuredangle DMT$. Furthermore, we have $\measuredangle DMT = \measuredangle TMF$ and $\measuredangle DMF = 2\measuredangle DMT = 2\measuredangle DAM$. In the circumcircle of DFD' with center M we obtain

$$\measuredangle DD'F = \frac{1}{2}\measuredangle DMF = \measuredangle DAM \quad .$$

We denote by I the intersection point of AM and $D'F$. From $\measuredangle DAI = \measuredangle DAM = \measuredangle DD'F = \measuredangle DD'I$ it follows that A, D, I, D' are concyclic.

In the cyclic quadrilateral $ADID'$ with chord AD' we obtain $90° = \measuredangle D'DA = \measuredangle D'IA$, i.e $AM \perp D'F$. Since $MF = MD'$ it follows that AM is the perpendicular bisector of FD' and we have $AF = AD'$, as required.



Solution 2. As shown in the first solution we only have to prove, that $AF = AD'$, where D' is the reflection of D with respect to the perpendicular bisector of the line segment BC.

In the triangle ADM we denote the projection of the point M onto the line AD by S and $\measuredangle DAM = \alpha$ and $\measuredangle MDA = \delta$. In the rectangular triangles ASM and MSD the following relations hold:

$$\begin{aligned} AM^2 = AS^2 + SM^2 \quad , \quad AS = AM\cos(\alpha) \quad , \quad SM = AM\sin(\alpha) \\ DM^2 = DS^2 + SM^2 \quad , \quad DS = DM\cos(\delta) \quad , \quad SM = DM\sin(\delta) \quad . \end{aligned} \tag{1}$$

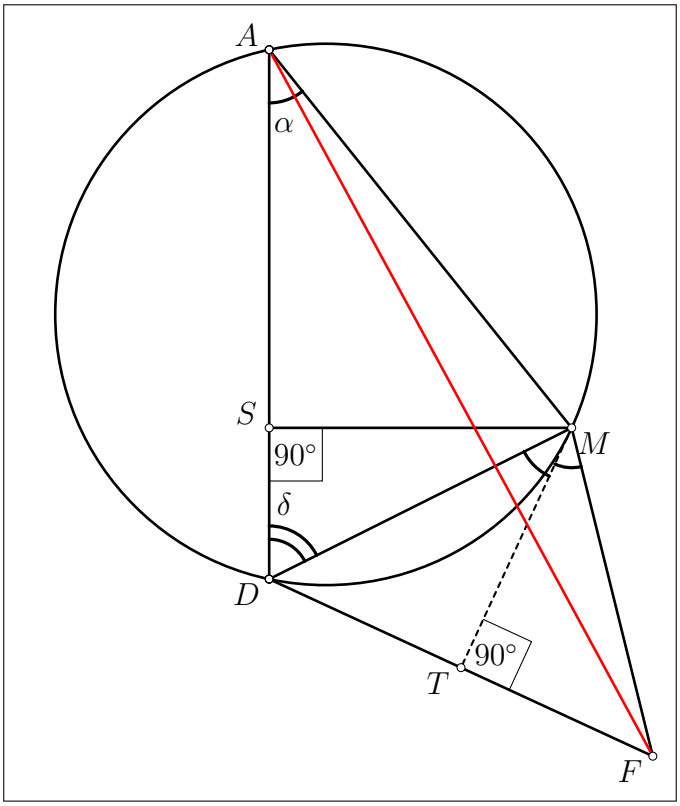
Let T be the center of the line segment DF. By definition of the point F we have $MD = MF$ and $\measuredangle DMT = \measuredangle TMF$ and the line TM is the tangent to the circumcircle of ADM at M. From the tangent chord angle theorem with chord DM in the circumcircle of ADM we obtain $\alpha = \measuredangle DMT$. It follows

$$\begin{aligned} \measuredangle AMF &= \measuredangle AMD + \measuredangle DMT + \measuredangle TMF \\ &= (180^\circ - \delta - \alpha) + 2\alpha \\ &= 180^\circ - \delta + \alpha \quad . \end{aligned} \tag{2}$$

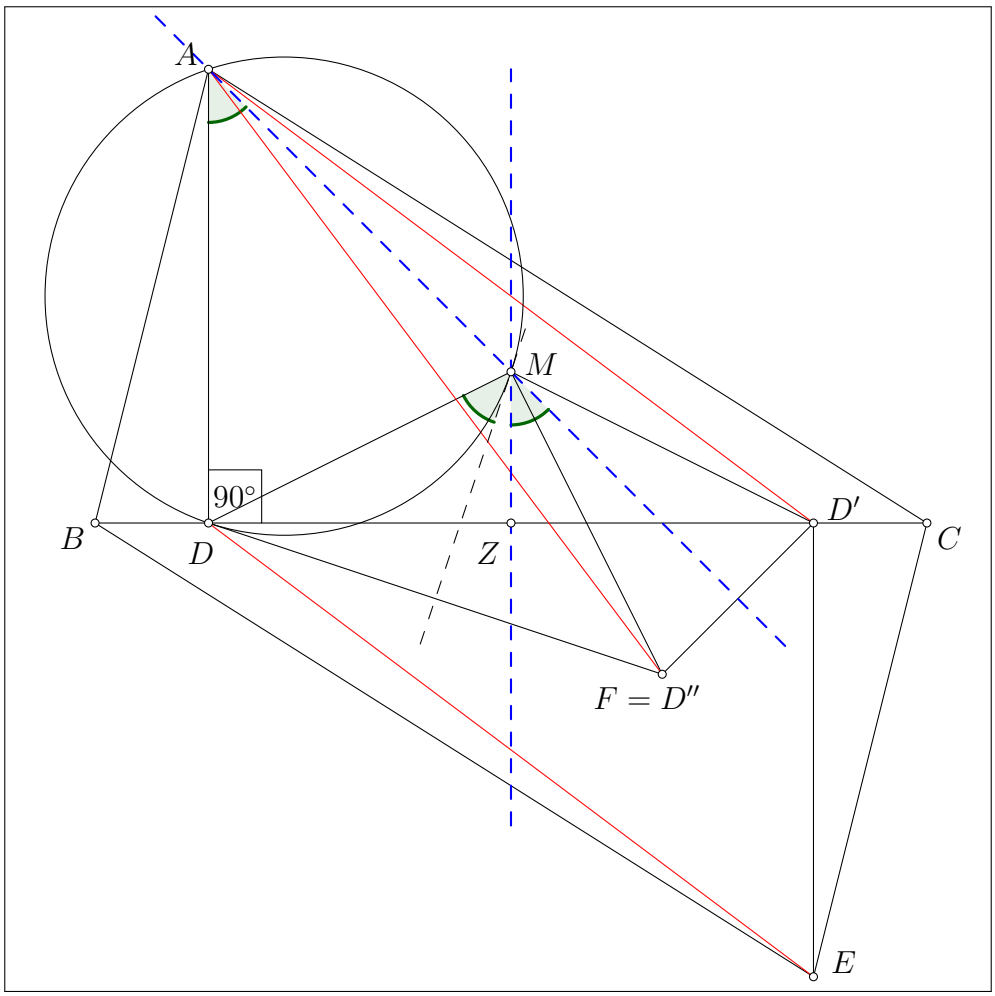
Using the law of cosine in the triangle AFM we obtain

$$\begin{aligned} AF^2 &= AM^2 + MF^2 - 2AM \cdot MF \cdot \cos(\measuredangle AMF) \\ &= AM^2 + MD^2 + 2AM \cdot MD \cdot \cos(-\delta + \alpha) \\ &= (AS^2 + SM^2) + (SD^2 + SM^2) + 2AM \cdot MD\,(\cos(\alpha)\cos(\delta) + \sin(\alpha)\sin(\delta)) \\ &= AS^2 + SD^2 + 2AS \cdot SD + 4SM^2 = AD^2 + 4SM^2 \quad . \end{aligned} \tag{3}$$

Since the point M is on the perpendicular bisector of the line segment $D'D$ and MS is parallel to DD' we have $DD' = 2MS$. Finally we get $AF^2 = AD^2 + (2SM)^2 = AD^2 + D'D^2 = AD'^2$, as required.



Solution 3. Let Z be the midpoint of BC. By definition of E, the point reflection in Z maps E to A; let it map D to D'. By the choice of M, MZ is the perpendicular bisector of BC, so the reflection across MZ also maps D to D'. Consider the concatenation with the reflection across AM which maps D' to some point D''. This concatenation is a rotation about the common point M of both axes by $2\angle(MZ, AM) = 2\angle DAM = \angle DMF$, since $DA \parallel MZ$ and $\angle DMF$ is by construction twice the tangent angle of DM at M. So D is mapped to F, i.e. $F = D''$, and we get $|AF| = |AD''| = |AD'| = |ED|$.


A
M
90°
B
D
Z
D′
C
F = D″
E

T-6

Let ABC be an acute triangle with an interior point D such that $\angle BDC = 180° - \angle BAC$. The lines BD and AC intersect at the point E, and the lines CD and AB intersect at the point F. The points $P \neq E$ and $Q \neq F$ lie on the line EF so that $BP = BE$ and $CQ = CF$. Assume that the segments AP and AQ intersect the circumcircle ω of ABC at the points $R \neq A$ and $S \neq A$, respectively. Prove that the lines RF and SE intersect on ω.

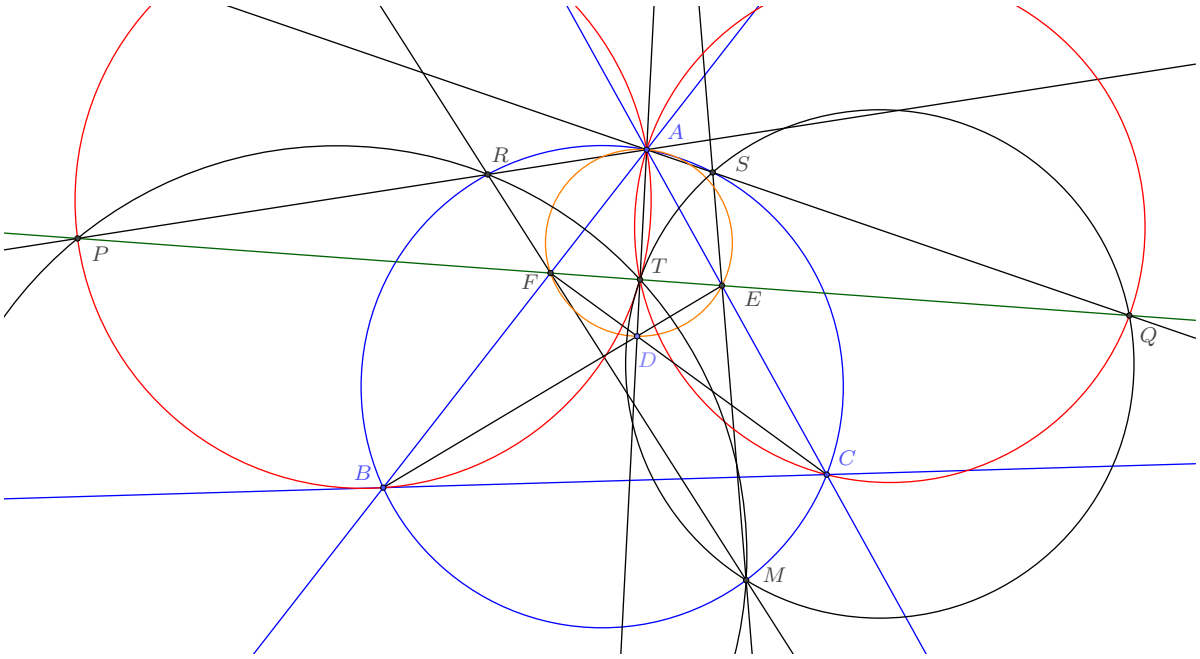
Solution 1. First of all, points A, F, D, E are concyclic due to $\angle EDF = \angle BDC = 180° - \angle BAC$.

Denote by T the intersection point of segments AD and EF. We will prove that $ATBP$ is a cyclic quadrilateral, by angle chasing:

$$\angle BAT = \angle FAD = \angle FED = \angle PEB = \angle BPE = \angle BPT.$$

Analogously, $ATCQ$ is a cyclic quadrilateral.

Consider triangle APQ and points R, S, T on its sides. Miquel's theorem gives that there is a common point M of the circumcircles of ARS, PTR, and QST.



Let us look at the cyclic quadruples (P, R, T, M), (A, T, B, P), and (A, R, B, M). Due to the radical axis theorem, the lines PT, AB, and RM are concurrent, which proves that the points R, F, M are collinear.

Analogously, the cyclic quadruples (Q, S, T, M), (A, T, C, Q), and (A, S, C, M) give that points S, E, M are collinear, which finishes the proof.

Solution 2. Let X be the intersection of BP and CQ.

Claim 1. X is on ω.
Because the triangles PBE and FCQ are isosceles we have $\measuredangle BPE = \measuredangle PEB$ and $\measuredangle CFQ = \measuredangle FQC$. It follows in the triangles PXQ and FDE

$$\measuredangle QXP = 180^\circ - \measuredangle BPE - \measuredangle FQC = 180^\circ - \measuredangle PEB - \measuredangle CFQ = \measuredangle EDF = 180^\circ - \measuredangle BAC.$$

If X and A lie on opposite sides of line BC, then the claim 1 follows directly. If X and A lie on the same side of line BC, then it follows that $\measuredangle BXC = 180^\circ - \measuredangle QXP = \measuredangle BAC$ and finally claim 1.

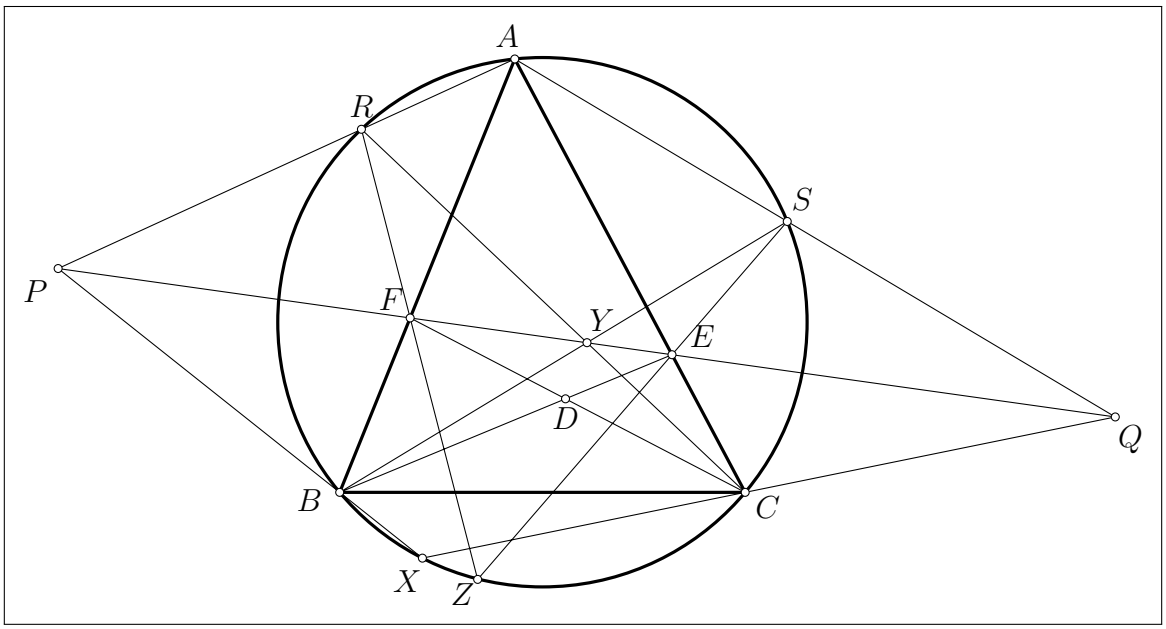
Let Y be the intersection of BS and CR.

Claim 2. The point Y is on the line $PQ = EF$.
By Pascal's theorem applied to $BSARCX$ the points $BS \cap CR = Y$, $SA \cap CX = Q$ and $AR \cap XB = P$ are on one line, which proves claim 2.

Let $Z \neq S$ be the second intersection of SE with ω.

Claim 3. Z is on the line RF.
By Pascal's theorem applied to $BSZRCA$ the points $BS \cap RC = Y$, $SZ \cap CA = E$ and $ZR \cap AB$ are on one line. Because of claim 2 Y is on EF and therefore the point $ZR \cap AB$ is also on EF, i.e. $F = ZR \cap AB$ and R, F and Z are on one line.

It follows that Z is the intersection of RF and SE which is also on ω.



Sketch of a solution using spiral similarity Let $X = BP \cap CQ$. As in the second solution, we have $X \in \omega$. Now let $G \neq X$ be the second intersection of circles $(XBC) = \omega$ and (XPQ).

Claim. *We have* $\triangle GFE \sim \triangle GRS$.

Proof. By the spiral similarity lemma, $\triangle GBC \sim \triangle GPQ$. So

$$\angle GPE = \angle GPQ = \angle GBC = \angle GAC = \angle GAE,$$

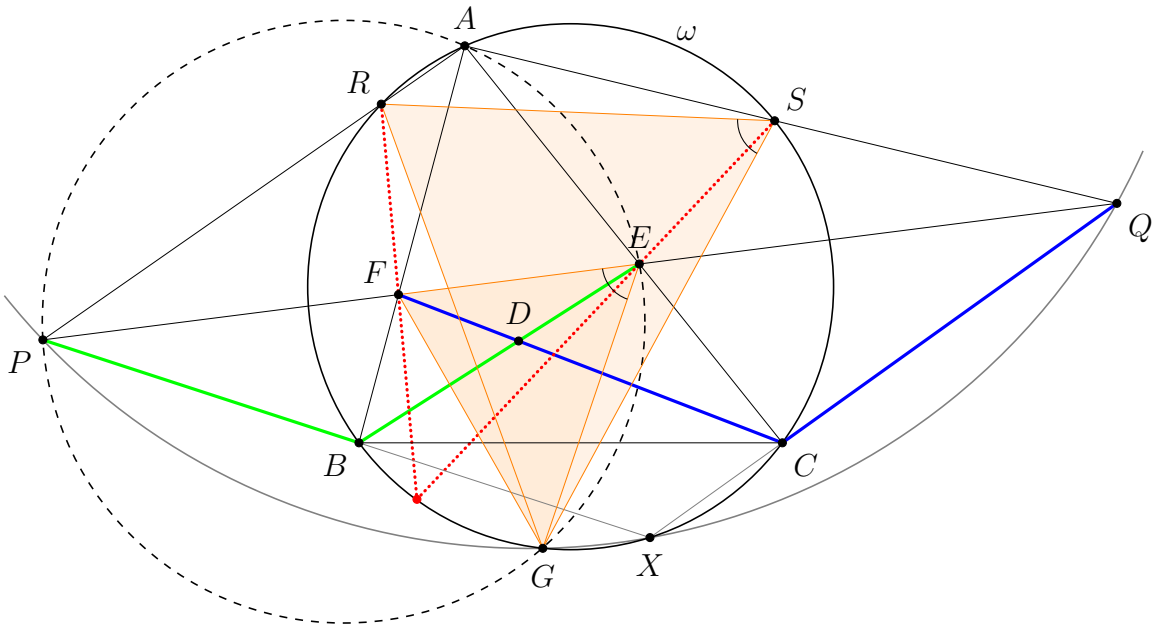
implying that $GEAP$ is a cyclic quadrilateral. This yields

$$\angle FEG = \angle PEG = \angle PAG = \angle RAG = \angle RSG.$$

Similarly, $GFAQ$ is cyclic and we obtain $\angle GFE = \angle GRS$. Therefore, $\triangle GFE \sim \triangle GRS$. □

From the claim, the spiral similarity lemma yields that $RF \cap SE$ lies on $(GRS) = \omega$.

Remark. Point G satisfies $(A, G; B, C) = -1$ and is in particular fixed when D moves.



T-7

Let n be a positive integer such that the *sum* of positive divisors of n^2+n+1 is divisible by 3. Prove that it is possible to partition the set of positive divisors of n^2+n+1 into three sets such that the *product* of all elements in each set is the same.

Solution. We first prove the following lemma.

Lemma. *Let p be prime divisor of n^2+n+1. Then $p=3$ or $p \equiv 1 \pmod 3$.*

Proof. We have $n^2+n+1 \equiv 0 \pmod p$. Multiplying by $n-1$, we get $n^3 \equiv 1 \pmod p$. Thus, the order of n modulo p is either 1 or 3. If the order is 1 then $n \equiv 1 \pmod p$, hence $0 \equiv n^2+n+1 \equiv 3 \pmod p$, and $p=3$. Otherwise, the order of n modulo p, which must divide $p-1$ according to Fermat, is 3. Therefore, $p \equiv 1 \pmod 3$. □

We now come back to the original problem.

Let $m := n^2+n+1$ and consider the prime decomposition $m = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ of m. Then the sum of the divisors of m is

$$\sigma(m) = \sigma(p_1^{\alpha_1}) \cdot \ldots \cdot \sigma(p_k^{\alpha_k}) = (1+p_1+p_1^2+\cdots+p_1^{\alpha_1}) \cdot \ldots \cdot (1+p_k+p_k^2+\cdots+p_1^{\alpha_k}) \,.$$

The assumption implies that at least one of the factors on the right hand side is divisible by 3.

If $p_i = 3$ then $(1+p_i+p_i^2+\cdots+p_i^{\alpha_i})$ is not divisible by 3. Hence, there must be some prime factor $p_j \equiv 1 \pmod 3$ such that $3 \mid (1+p_j+p_j^2+\cdots+p_j^{\alpha_j})$, which shows $\alpha_j \equiv 2 \pmod 3$. The number of divisors of m is $d(m) = d(p_1^{\alpha_1}) \cdot \ldots \cdot d(p_k^{\alpha_k}) = (\alpha_1+1) \cdot \ldots \cdot (\alpha_k+1)$. Since $\alpha_j \equiv 2 \pmod 3$, one may conclude that the number of divisors of m is divisible by 3, too.

Moreover, one observes that m is not a perfect square. Indeed, $n^2 < m = n^2+n+1 < (n+1)^2$. Let d be any divisor of m. Then $\frac{m}{d}$ is also a divisor of n, and $d \neq \frac{m}{d}$. Thus, one may pair every divisor d of m with the complementary divisor $\frac{m}{d}$. The product of the elements of any of these pairs is then equal to m

Since the number of divisors is divisible by 3, the same is true for the number of pairs. One can therefore split the set of these pairs into three subsets such that each subset consists of the same number of pairs. This leads to three groups of divisors, and the product of all elements of each group must be the same.

Remarks: 1. There are other, less elegant ways to describe a partition of the divisors of n^2+n+1 into three groups such that each group has the same product. Let us indicate at least one of them.

Write $m = p_j^{\alpha_j} \cdot N$, for $N := m/p_j^{\alpha_j}$, and suppose that $\alpha_j \equiv 2 \pmod 3$. Then every divisor of m may uniquely be written in the form $p_j^a \cdot n$, for $0 \le a \le \alpha_j$ and $n \mid N$. Let us distinguish between two cases.

$\alpha_j \equiv 5 \pmod 6$.

Then write $\alpha_j = 6l - 1$, for a positive integer l, and put

$$G_1 := \{p_j^a \cdot n \mid 0 \le a \le l-1 \text{ or } 5l \le a \le 6l-1\},$$
$$G_2 := \{p_j^a \cdot n \mid l \le a \le 2l-1 \text{ or } 4l \le a \le 5l-1\}, \text{ and}$$
$$G_3 := \{p_j^a \cdot n \mid 2l \le a \le 4l-1\}.$$

$\alpha_j \equiv 2 \pmod 6$.

Then N is a non-square. Put, at first,

$$G_1 := \{p_j^a \cdot n \mid a \equiv 1 \pmod 3\}.$$

The remaining divisors of m come in blocks of four, and are distributed among G_2 and G_3 according to the following rule. If $p_j^a \cdot n$ is put into G_2 then put $p_j^{\alpha_j - a} \cdot (N/n)$ into G_2, as well, but put $p_j^{\alpha_j - a} \cdot n$ and $p_j^a \cdot (N/n)$ into G_3.

2. The smallest positive integer n such that the sum of the positive divisors of $n^2 + n + 1$ is divisible by 3 is $n = 22$. Then

$$n^2 + n + 1 = 507 = 3 \cdot 13^2$$

and $\sigma(507) = (1+3) \cdot (1 + 13 + 13^2) \equiv 0 \pmod 3$.

T-8

Determine whether the following statement is true for every polynomial P of degree at least 2 with nonnegative integer coefficients:

There exists a positive integer m such that for infinitely many positive integers n the number $P^n(m)$ has more than n distinct positive divisors.

Remark. Here P^n denotes P applied n times, this means $P^n(x) = \underbrace{P(P(\ldots P}_{n \text{ times}}(x)\ldots))$.

Answer. Yes, there is always a positive integer m such that $P^n(m)$ has more than n distinct positive divisors for infinitely many positive integers n.

Solution. We will show that

$$m = \begin{cases} 2\,, & \text{for } P(0) = 0\,, \\ P(0)\,, & \text{for } P(0) > 0\,, \end{cases}$$

works.

Let $\tau(r)$ denote the number of distinct positive divisors of the positive integer r.

Case 1: $P(0) = 0$.

Since the polynomial P has degree at least two and non-negative coefficients, one has $P(\ell) \geq \ell^2 > 2$, for every $\ell \geq 2$. In particular, one may conclude inductively that

$$P^k(2) > P^{k-1}(2) > 2$$

holds, for every integer $k \geq 2$.

Furthermore, $P(0) = 0$ implies $x \mid P(x)$, whence each divisor of $P^{k-1}(2)$ is a divisor of $P^k(2)$, too. Together with the fact that $P^k(2) > P^{k-1}(2)$ has at least one further divisor (itself), one knows that $\tau(P^k(2)) \geq \tau(P^{k-1}(2)) + 1$, for every integer $k \geq 2$. From this, by induction, one gets

$$\tau\big(P^k(2)\big) \geq \tau\big(P(2)\big) + k - 1 \geq 2 + k - 1 > k\,,$$

as asserted.

Case 2: $P(0) > 0$.

The following lemma is essential.

Lemma. *Let A and B be integers such that $A \mid B$ and $B > A^2$. Then*

$$\tau(B) \geq 2\tau(A)\,.$$

Proof. One has $B > A^2 \geq 1$, so that B has at least one prime factor. Thus, one can write $B = p_1^{b_1} p_2^{b_2} \ldots p_i^{b_i}$, for distinct primes p_j and positive integers b_j. Moreover, since $A \mid B$, one has $A = p_1^{a_1} p_2^{a_2} \ldots p_i^{a_i}$, for non-negative integers a_j such that $a_j \leq b_j$.

The assumption $B > A^2$ implies the existence of an index k such that $b_k > 2a_k$. Since

$$\tau(B) = \prod_{j=1}^{i} (b_j + 1) \qquad \text{and} \qquad \tau(A) = \prod_{j=1}^{i} (a_j + 1)\,,$$

it turns out that indeed

$$\frac{\tau(B)}{\tau(A)} = \prod_{j=1}^{i} \frac{b_j + 1}{a_j + 1} \geq \frac{b_k + 1}{a_k + 1} \geq \frac{(2a_k + 1) + 1}{a_k + 1} = 2\,. \qquad \square$$

Corollary. Let $Q(x)$ be a polynomial with non-negative coefficients and degree at least 2, such that $Q(0) > 0$. Then

$$\tau\big(Q\big(Q(0)\big)\big) \geq 2\tau\big(Q(0)\big).$$

Proof. Since Q has integer coefficients, $(a - b) \mid \big(Q(a) - Q(b)\big)$, for any integers a and b. In particular, $\big(Q(0) - 0\big) \mid \big(Q\big(Q(0)\big) - Q(0)\big)$, so $Q(0) \mid \big(Q\big(Q(0)\big) - Q(0)\big)$ and

$$Q(0) \mid Q\big(Q(0)\big)\,.$$

If we write $Q(x) = a_d x^d + \ldots a_1 x + a_0$ then $a_0 = Q(0) > 0$, so

$$Q(x) \geq a_d x^d + a_0 > a_d x^d \geq x^d \geq x^2\,,$$

for any non-negative integer x. In particular, $Q\big(Q(0)\big) > Q(0)^2 \geq 1$.

Thus, the lemma applies with $A := Q(0)$ and $B := Q\big(Q(0)\big)$, and shows that indeed $\tau\big(Q\big(Q(0)\big)\big) \geq 2\tau\big(Q(0)\big)$. $\square$

Returning to the original problem, let us put $m := P(0)$. We will show by induction on k that, for all $n := 2^k - 1$, the value $P^n(m) = P^{2^k}(0)$ has at least $n + 1 = 2^k$ positive divisors.

For $k = 1$, the corollary implies that $P\big(P(0)\big)$ has at least 2 positive divisors, which is exactly what is stated.

Finally, when the statement is to be shown for some fixed $k \geq 2$, one may suppose that it is already known for $k - 1$. Then, using the induction hypothesis and the corollary for

$Q := P^{2^k}$, one gets

$$\tau\Big(P^{2^k-1}(m)\Big) = \tau\Big(P^{2^k}(0)\Big) = \tau\Big(P^{2^{k-1}}\Big(P^{2^{k-1}}(0)\Big)\Big) \geq 2\tau\Big(P^{2^{k-1}}(0)\Big) \geq 2 \cdot 2^{k-1} = 2^k\,.$$

This completes the solution.