Problem I–1

Let $\mathbb{R}^+$ be the set of positive real numbers. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$ it holds that

$$y f^{2025}(x) \geq x f(y)\,.$$

Show that there exists a positive integer n_0 such that for all positive integers $n \geq n_0$ and for all $x \in \mathbb{R}^+$ it holds that

$$f^n(x) \geq x.$$

Remark. Here f^n denotes the function f applied n times, this means $f^n(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots))$.

Problem I–2

On an infinite square grid, on which some unit squares are coloured red, a *ruby rook* is a piece which, in one move, can travel any number of squares in one direction parallel to one of the grid lines (either vertically or horizontally), while remaining on red squares at all times throughout the move.

Starting with an uncoloured infinite square grid, Alice performs the following procedure: First, she colours at most 2025 of the unit squares red. Afterwards, she places some ruby rooks on distinct red unit squares, such that the following two rules are satisfied:

- No ruby rook can reach another ruby rook in one move.
- Every ruby rook can reach every other ruby rook in two moves.

Find the maximum possible number of ruby rooks that Alice can place during this procedure.

Problem I–3

Let ABC be a triangle. Its incircle ω touches the sides BC, CA and AB at points D, E and F, respectively. Let P and Q be points on the line BC distinct from D such that $PB = BD$ and $QC = CD$. Prove that the circumcircles of the triangles PCE and QBF and the circle ω pass through a common point.

Problem I–4

A subset S of the integers is called *Saxonian* if for every three pairwise different elements a, b, $c \in S$ the number $ab + c$ is the square of an integer. Prove that any Saxonian set is finite. Determine the largest possible number of elements that a Saxonian set can have.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.